

LOCAL PROPERTIES OF QUASIHYPERSBOLIC AND FREELY QUASICONFORMAL MAPPINGS

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ABSTRACT. Suppose that E and E' denote real Banach spaces with dimension at least 2, that $D \subset E$ and $D' \subset E'$ are domains, and that $f : D \rightarrow D'$ is a homeomorphism. In this paper, we prove that if there exists some constant $M > 1$ (resp. some homeomorphism φ) such that for all $x \in D$, $f : B(x, d_D(x)) \rightarrow f(B(x, d_D(x)))$ is M -QH (resp. φ -FQC), then f is M_1 -QH with $M_1 = M_1(M)$ (resp. φ_1 -FQC with $\varphi_1 = \varphi_1(\varphi)$). We apply our results to establish, in terms of the j_D metric, a sufficient condition for a homeomorphism to be FQC.

1. INTRODUCTION AND MAIN RESULTS

During the past few decades, modern mapping theory and the geometric theory of quasiconformal maps have been studied from several points of view. These studies include Heinonen's work on metric measure spaces [5], Koskela's study of maps with finite distortion [7] and Väisälä's work dealing with quasiconformality in infinite dimensional Banach spaces [15, 16, 17, 18, 19]. The quasihyperbolic metric is an important tool in each of these investigations although their respective methods are otherwise quite divergent. In this paper we will study some questions left open by Väisälä's work. In passing we remark that the quasihyperbolic geometry has been recently studied by many people(cf. [4, 6, 8, 10]).

Throughout the paper, we always assume that E and E' denote real Banach spaces with dimension at least 2, and that $D \subset E$ and $D' \subset E'$ are domains. The norm of a vector z in E is written as $|z|$, and for each pair of points z_1, z_2 in E , the distance between them is denoted by $|z_1 - z_2|$, the closed line segment with endpoints z_1 and z_2 by $[z_1, z_2]$. The distance from $z \in D$ to the boundary ∂D of D is denoted by $d_D(z)$. For an open ball with center x and radius r we use the notation $\mathbb{B}(x, r)$. The boundary sphere is denoted by $\mathbb{S}(x, r)$. We begin with the following concepts in line with the notation and terminology of [11, 13, 14, 15, 16].

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Let X be a metric space and $\dot{X} = X \cup \{\infty\}$. By a triple in X we mean an ordered sequence $T = (x, a, b)$ of three distinct points in X . The ratio of T is the number

$$\rho(T) = \frac{|a - x|}{|b - x|}.$$

If $f : X \rightarrow Y$ is an injective map, the image of a triple $T = (x, a, b)$ is the triple $fT = (fx, fa, fb)$.

Definition 1. Let X and Y be two metric spaces, and let $\eta : [0, \infty) \rightarrow [0, \infty)$ be a homeomorphism. Suppose $A \subset X$. An embedding $f : X \rightarrow Y$ is said to be η -quasisymmetric or briefly η -QS, if $\rho(T) \leq t$ implies that $\rho(f(T)) \leq \eta(t)$ for each triple T in X and $t \geq 0$. We note that $\eta(1) \geq 1$ always holds.

For convenience, in what follows, we always assume that x, y, z, \dots denote points in D and x', y', z', \dots the images in D' of x, y, z, \dots under f , respectively. Also we assume that $\alpha, \beta, \gamma, \dots$ denote curves in D and $\alpha', \beta', \gamma', \dots$ the images in D' of $\alpha, \beta, \gamma, \dots$ under f , respectively.

Definition 2. Let $0 < q < 1$, and let D, D' be metric spaces in E and E' , respectively. A homeomorphism $f : D \rightarrow D'$ is q -locally η -quasisymmetric if $f|_{\mathbb{B}(a, qr)}$ is η -QS whenever $\mathbb{B}(a, r) \subset D$. If $D \neq E$, this means that $f|_{\mathbb{B}(a, qd_D(a))}$ is η -QS. When $D = E$, obviously, f is η -QS.

Definition 3. A map $f : X \rightarrow Y$ is uniformly continuous if and only if there is $t_0 \in (0, \infty]$ and an embedding $\varphi : [0, t_0) \rightarrow [0, \infty)$ with $\varphi(0) = 0$ such that

$$|f(x) - f(y)| \leq \varphi(|x - y|)$$

whenever $x, y \in X$ and $|x - y| \leq t_0$. We then say that f is (φ, t_0) -uniformly continuous. If $t_0 = \infty$, we briefly say that f is φ -uniformly continuous.

The definitions of k_D and j_D metric will be given in section 2.

Definition 4. Let $D \neq E$ and $D' \neq E'$ be metric spaces, and let $\varphi : [0, \infty) \rightarrow [0, \infty)$ be a growth function, that is, a homeomorphism with $\varphi(t) \geq t$. We say that a homeomorphism $f : D \rightarrow D'$ is φ -semisolid if

$$k_{D'}(f(x), f(y)) \leq \varphi(k_D(x, y))$$

for all $x, y \in D$, and φ -solid if both f and f^{-1} satisfy this condition.

The special case $\varphi(t) = Mt$, $M \geq 1$, gives the M -quasihyperbolic maps or briefly M -QH. More precisely, f is called M -QH if

$$\frac{k_D(x, y)}{M} \leq k_{D'}(f(x), f(y)) \leq Mk_D(x, y)$$

for all x and y in D .

We say that f is *fully φ -semisolid* (resp. *fully φ -solid*) if f is φ -semisolid (resp. φ -solid) on every subdomain of D . In particular, when $D = E$, the subdomains are taken to be proper ones in D . Fully φ -solid mappings are also called *freely φ -quasiconformal mappings*, or briefly *φ -FQC mappings*.

If $E = \mathbb{R}^n = E'$, then f is FQC if and only if f is quasiconformal (cf. [15]). See [12, 21] for definitions and properties of K -quasiconformal mappings, or briefly K -QC mappings. It is known that each K -QC mapping in \mathbb{R}^n is q -locally η -QS for every $q < 1$ with $\eta = \eta(K, q, n)$, i.e. η depends only on the constants K , q and n (cf. [1, 5.23]). Conversely, each q -locally η -QS mapping in \mathbb{R}^n is a K -QC mapping with $K = (\eta(1))^{n-1}$ by the metric definition of quasiconformality (cf. [15, 5.6]). Further, in [15], Väisälä proved

Theorem A. ([15, Theorem 5.10]) *For a homeomorphism $f : D \rightarrow D'$, the following conditions are quantitatively equivalent:*

- (1) f is φ - FQC ;
- (2) for some fixed $q \in (0, 1)$, both f and f^{-1} are locally η -QS;
- (3) For every $0 < q < 1$, there is some η_q such that both f and f^{-1} are q -locally η_q -QS.

For M -QH mappings, Väisälä [15, 19] proved the following.

Theorem B. ([15, Theorem 4.7]) *Suppose that $D \neq E$, $D' \neq E'$ and that $f : D \rightarrow D'$ is M -QH. Then f is fully $4M^2$ -QH.*

Theorem C. ([19, Theorem 5.16]) *Suppose that $f : D \rightarrow D'$ is a homeomorphism and that each point has a neighborhood $D_1 \subset D$ such that $f_{D_1} : D_1 \rightarrow f(D_1)$ is M -bilipschitz (or briefly f is locally M -bilipschitz). Then f is M^2 -QH.*

Recall that M -QH need not be bilipschitz. Hence the following problem of Väisälä is natural.

Open Problem 1. ([19, Section 13]) *Suppose that $f : D \rightarrow D'$ is a homeomorphism and that there exists $M > 1$ such that for each point has a neighborhood $D_1 \subset D$ such that $f_{D_1} : D_1 \rightarrow f(D_1)$ is M -QH. Is f M' -QH with $M' = M'(M)$?*

The first aim of this paper is to study Open Problem 1. Our result is as follows.

Theorem 1. *Suppose that $f : D \rightarrow D'$ is a homeomorphism and there exists some constant $M > 1$ such that for each point $x \in D$, $f : \mathbb{B}(x, d_D(x)) \rightarrow f(\mathbb{B}(x, d_D(x)))$ is M -QH. Then f is M_1 -QH with $M_1 = M_1(M)$.*

Further, in [15], Väisälä raised the following open problem.

Open Problem 2. ([15, Section 7]) *Suppose that $f : G \rightarrow G'$ is a homeomorphism and that each point has a neighborhood $D \subset G$ such that $f_D : D \rightarrow f(D)$ is φ - FQC . Is f φ' - FQC with $\varphi' = \varphi'(\varphi)$?*

The second aim of this paper is to consider Open Problem 2. By applying Theorem 1, we will prove the following theorem.

Theorem 2. *Suppose that $f : D \rightarrow D'$ is a homeomorphism and there exists some homeomorphism φ such that for each point $x \in D$, $f : \mathbb{B}(x, d_D(x)) \rightarrow f(\mathbb{B}(x, d_D(x)))$ is φ - FQC . Then f is φ_1 - FQC with $\varphi_1 = \varphi_1(\varphi)$.*

Applying Theorem 2 we prove the following.

Theorem 3. *Let $\varphi : [0, \infty) \rightarrow [0, \infty)$ be homeomorphism with $\varphi(t) \geq t$ for all t . Suppose that $f : D \rightarrow D'$ is a homeomorphism and that for every subdomain $D_1 \subset D$, we have*

$$(1.1) \quad \varphi^{-1}(j_{D_1}(x, y)) \leq j_{D'_1}(x', y') \leq \varphi(j_{D_1}(x, y))$$

with $x, y \in D_1$. Then f is φ_1 -FQC with $\varphi_1 = \varphi_1(\varphi)$.

The following two examples show that the converse of Theorem 3 is not true.

Example 1.2. Let $E = \mathbb{R}^2$ and $f : D = \mathbb{B}(0, 1) \rightarrow D' = \mathbb{B}(0, 1) \setminus [0, 1)$ be a conformal mapping. There exist points $x, y \in D$ such that (1.1) does not hold.

Example 1.3. We consider the broken tube 4.12 in [15]. Let E be an infinite-dimensional separable Hilbert space, and choose an orthonormal base $(e_j)_{j \in \mathbf{Z}}$ of E indexed by the set \mathbf{Z} of all integers. Setting $\gamma'_j = [e_{j-1}, e_j]$ we obtain the infinite broken line $\gamma' = \cup\{\gamma'_j : j \in \mathbf{Z}\}$. Let γ denote the line spanned by e_1 , and let $D = \gamma + \mathbb{B}(r)$ with $r \leq \frac{1}{10}$ and f be a locally M -bilipschitz homeomorphism from D onto a neighborhood D' of γ' (For more detail see [15]). There exist points $x, y \in D$ such that (1.1) does not hold.

The proofs of Theorem 1 and Theorems 2 will be given in Section 3. The proofs of Theorem 3 and Examples 1.2 and 1.3 will be given in Section 4. In Section 2, some necessary preliminaries will be introduced.

2. PRELIMINARIES

The *quasihyperbolic length* of a rectifiable arc or a path α in the norm metric in D is the number (cf. [3, 20]):

$$\ell_k(\alpha) = \int_{\alpha} \frac{|dz|}{d_D(z)}.$$

For each pair of points z_1, z_2 in D , the *quasihyperbolic distance* $k_D(z_1, z_2)$ between z_1 and z_2 is defined in the usual way:

$$k_D(z_1, z_2) = \inf \ell_k(\alpha),$$

where the infimum is taken over all rectifiable arcs α joining z_1 to z_2 in D . For each pair of points z_1, z_2 in D , we have (cf. [20])

$$(2.1) \quad k_D(z_1, z_2) \geq \log \left(1 + \frac{|z_1 - z_2|}{\min\{d_D(z_1), d_D(z_2)\}} \right) \doteq j_D(z_1, z_2).$$

Gehring and Palka [3] introduced the quasihyperbolic metric of a domain in \mathbb{R}^n . Many of the basic properties of this metric may be found in [2]. Recall that an arc α from z_1 to z_2 is a *quasihyperbolic geodesic* if $\ell_k(\alpha) = k_D(z_1, z_2)$. Each subarc of a quasihyperbolic geodesic is obviously a quasihyperbolic geodesic. It is known that a quasihyperbolic geodesic between every pair of points in E exists if the dimension of E is finite, see [2, Lemma 1]. This is not true in arbitrary spaces (cf. [18, Example

2.9)). In order to remedy this shortage, Väisälä introduced the following concepts [16].

Definition 5. Let $D \neq E$ and $c \geq 1$. An arc $\alpha \subset D$ is a c -neargeodesic if and only if $\ell_k(\alpha[x, y]) \leq c k_D(x, y)$ for all $x, y \in \alpha$.

In [18], Väisälä proved the following property concerning the existence of near-geodesics in E .

Theorem D. ([18, Theorem 3.3]) *Let $\{z_1, z_2\} \subset D$ and $c > 1$. Then there is a c -neargeodesic in D joining z_1 and z_2 .*

Definition 6. We say that a homeomorphism $f : D \rightarrow D'$ is C -coarsely M -quasihyperbolic, or briefly (M, C) -CQH, in the quasihyperbolic metric if it satisfies

$$\frac{k_D(x, y) - C}{M} \leq k_{D'}(f(x), f(y)) \leq M k_D(x, y) + C$$

for all $x, y \in D$, where D and D' are domains in E and E' , respectively.

The following result was proved by Väisälä in [17].

Theorem E. ([17, Corollary 2.21]) *Suppose that $D \subset E$ and $D' \subset E'$ are domains and that $f : D \rightarrow D'$ is a homeomorphism. Then the following conditions are quantitatively equivalent:*

- (1) f is φ -FQC;
- (2) f is (M, C) -CQH in every subdomain of D ,

where φ and (M, C) depend only on each other.

Definition 7. Let $0 < t_0 \leq 1$ and let $\theta : [0, t_0) \rightarrow [0, \infty)$ be an embedding with $\theta(0) = 0$. Suppose that $f : D \rightarrow D'$ is a homeomorphism with $D \neq E$ and $D' \neq E'$. We say that the homeomorphism f is (θ, t_0) -relative if

$$\frac{|f(x) - f(y)|}{d_{D'}(f(x))} \leq \theta\left(\frac{|x - y|}{d_D(x)}\right)$$

for all $x, y \in G$ and $|x - y| < t_0 d_D(x)$. If $t_0 = 1$, then we say that f is θ -relative.

Concerning the relations between relative homeomorphisms and solid mappings, Väisälä proved the following.

Theorem F. ([15, Corollary 3.8]) *Suppose that $D \neq E$, $D' \neq E'$ and $f : D \rightarrow D'$ is a homeomorphism. Then the following conditions are quantitatively equivalent:*

- (1) f and f^{-1} are θ -relative;
- (2) f and f^{-1} are (θ, t_0) -relative;
- (3) f is φ -solid.

The following lemma is useful for the proof of Theorem 1.

Lemma 1. *Let $D \subset E$ be a domain, and let $x \in D$ and $0 < s < 1$. If $|x - y| \leq sd_D(x)$, then*

$$k_{B_x}(x, y) \leq \frac{1}{1-s} \log \left(1 + \frac{|x - y|}{d_D(x)} \right),$$

where $B_x = \mathbb{B}(x, d_D(x))$.

Proof. For each $w \in [x, y]$, we have $d_{B_x}(w) \geq d_D(x) - |x - w|$. Then by the Bernoulli inequality

$$\begin{aligned} k_{B_x}(x, y) &\leq \int_{[x, y]} \frac{|dw|}{d_{B_x}(w)} \leq \int_{d_D(x) - |x - y|}^{d_D(x)} \frac{dt}{t} \\ &= \log \left(1 + \frac{|x - y|}{d_D(x) - |x - y|} \right) \leq \log \left(1 + \frac{|x - y|}{(1-s)d_D(x)} \right) \\ &\leq \frac{1}{1-s} \log \left(1 + \frac{|x - y|}{d_D(x)} \right), \end{aligned}$$

from which our lemma follows. \square

We remark that when $E = R^n$, Lemma 1 coincides with Lemma 3.7 in [21].

3. THE PROOFS OF THEOREMS 1 AND 2

3.1. The proof of Theorem 1. For fixed $x \in D$, we let $D_1 = \mathbb{B}(x, d_D(x))$. Then it follows from Theorem A that there exist some constant $c \geq 2$ such that $f : D_1 \rightarrow D'_1$ is $\frac{c(c+1)}{1+c(c+1)}$ -locally η -QS with $\eta(\frac{1}{c}) < 1$ and $\eta = \eta(c, M)$. Theorem A shows that $f : D_1 \rightarrow D'_1$ is also a $(1 - \frac{1}{c^4})$ -locally η_1 -QS mapping with $\eta_1 = \eta_1(M)$. Furthermore, we infer from Theorem F that $f : D_1 \rightarrow D'_1$ is θ -relative with $\theta = \theta(M)$. Let $q = \frac{c(c+1)}{1+c(c+1)}$, and let $b_1 = 4M^2(\theta(q) + \eta_1(1) + \frac{\theta(q)}{1-\eta(\frac{1}{c})})$.

For a fixed $x \in D$, let $w_1 \in \mathbb{S}(x, d_D(x)) \cap \partial D$. We have the following lemma.

Lemma 2. $d_{D'_1}(x') \geq \frac{1}{b_1} d_{D'}(x')$.

Proof. Suppose on the contrary that

$$(3.1) \quad d_{D'_1}(x') < \frac{1}{b_1} d_{D'}(x').$$

Because $f : D_1 \rightarrow D'_1$ is θ -relative, it follows that for all $z \in \mathbb{S}(x, qd_{D_1}(x))$,

$$\frac{|x' - z'|}{d_{D'_1}(x')} \leq \theta(q).$$

Hence (3.1) shows

$$(3.2) \quad \text{diam} \left(f(\mathbb{S}(x, qd_{D_1}(x))) \right) \leq 2\theta(q) d_{D'_1}(x') \leq \frac{2\theta(q)}{b_1} d_{D'}(x').$$

Let $w'_2 \in \mathbb{S}(x', \frac{1}{2}d_{D'}(x')) \cap f([x, w_1])$ such that $[x, w_2) \subset f^{-1}\left(\mathbb{B}(x', \frac{1}{2}d_{D'}(x'))\right)$. We have the following claim.

Claim. $|w_1 - w_2| \geq \frac{1}{c^4}d_D(x)$.

We prove this claim also by contradiction. Suppose that

$$|w_1 - w_2| < \frac{1}{c^4}d_D(x).$$

Let w_3, w_4, \dots, w_m be the points in $[w_2, x]$ in the direction from w_2 to x such that for each $i \in \{3, \dots, m\}$

$$|w_i - w_{i-1}| = c|w_{i-1} - w_{i-2}| \text{ and } |w_m - x| < c|w_m - w_{m-1}|.$$

It is possible that $w_m = x$. Obviously, $k > 4$. Since

$$|w_1 - w_{m-1}| > |w_{m-2} - w_{m-1}| = \frac{1}{c}|w_m - w_{m-1}|$$

and

$$|w_{m-1} - x| = |w_{m-1} - w_m| + |w_m - x| < (c+1)|w_{m-1} - w_m|,$$

we see that

$$|w_1 - w_{m-1}| > \frac{1}{c(c+1)}|w_{m-1} - x|.$$

Then

$$|w_{m-1} - x| = |w_1 - x| - |w_1 - w_{m-1}| < |w_1 - x| - \frac{1}{c(c+1)}|w_{m-1} - x|,$$

which implies

$$|w_{m-1} - x| < \frac{c(c+1)}{1+c(c+1)}|w_1 - x| = \frac{c(c+1)}{1+c(c+1)}d_{D_1}(x).$$

Hence

$$w_{m-1} \in \mathbb{B}\left(x, \frac{c(c+1)}{1+c(c+1)}d_{D_1}(x)\right),$$

whence, by (3.2),

$$(3.3) \quad \max\{|w'_{m-1} - w'_m|, |w'_m - x'|\} < \frac{2\theta(q)}{b_1}d_{D'}(x').$$

For each $i \in \{3, \dots, m-1\}$, let $y_i = \frac{1}{2}(w_{i-1} + w_{i+1})$. Then

$$|w_1 - w_{i-1}| \geq \frac{1}{c}|w_{i-1} - w_i| = \frac{1}{2c}|w_{i-1} - y_i|,$$

which implies

$$|w_1 - w_{i-1}| \geq \frac{1}{2c+1}d_{D_1}(y_i),$$

whence

$$w_{i-1}, w_i, w_{i+1} \in \mathbb{B}\left(y_i, \frac{2c}{2c+1}d_{D_1}(y_i)\right) \subset \mathbb{B}(x, qd_{D_1}(y_i)).$$

Because f is q -locally η -QS it follows that

$$\frac{|w'_{i-1} - w'_i|}{|w'_i - w'_{i+1}|} \leq \eta \left(\frac{|w_{i-1} - w_i|}{|w_i - w_{i+1}|} \right) \leq \eta \left(\frac{1}{c} \right),$$

which together with the choice of w_2 and the inequality (3.3) imply that

$$\begin{aligned} \left(\frac{1}{2} - \frac{2\theta(q)}{b_1} \right) d_{D'}(x') &\leq |w'_2 - x'| - |x' - w'_m| \leq \sum_{i=3}^m |w'_i - w'_{i-1}| \\ &\leq |w'_{m-1} - w'_m| \sum_{i=0}^{m-3} \left(\eta \left(\frac{1}{c} \right) \right)^i \leq \frac{2\theta(q)}{(1 - \eta(\frac{1}{c}))b_1} d_{D'}(x') \\ &< \frac{1}{4} d_{D'}(x'). \end{aligned}$$

This obvious contradiction completes the proof of Claim.

Hence the above Claim shows

$$|x - w_2| = |x - w_1| - |w_1 - w_2| \leq \left(1 - \frac{1}{c^4} \right) d_{D_1}(x).$$

Let $y' \in \mathbb{S}(x', d_{D'_1}(x')) \cap D'_1$, and let $w_0 \in f^{-1}([x', y']) \cap \mathbb{S}(x, |x - w_2|)$. Because f is also $(1 - \frac{1}{c^4})$ -locally η_1 -QS, we see that

$$\frac{b_1}{2} \leq \frac{|x' - w'_2|}{|x' - w'_0|} \leq \eta_1(1).$$

This contradiction completes the proof of Lemma 2. \square

Now we are ready to finish the proof of Theorem 1.

To prove Theorem 1, we only need to prove that for $x, y \in D$, the following inequalities hold:

$$(3.4) \quad \frac{1}{M_1} k_D(x, y) \leq k_{D'}(x', y') \leq M_1 k_D(x, y).$$

We divide the discussions into two cases.

Case 1. $|x - y| \leq \frac{1}{2} d_D(x)$.

By Lemma 1 we have

$$(3.5) \quad k_{D'}(x', y') \leq k_{D'_1}(x', y') \leq M k_{D_1}(x, y) \leq 2M k_D(x, y).$$

If $|x' - y'| \leq \frac{1}{2} d_{D'_1}(x')$, we know from Theorem B that $f^{-1} : D'_2 = \mathbb{B}(x', d_{D'_1}(x')) \rightarrow f^{-1}(D'_2)$ is $4M^2$ -QH. Hence again by Lemma 1 and Lemma 2 we conclude that

$$\begin{aligned} (3.6) \quad k_D(x, y) &\leq k_{D_2}(x, y) \leq 4M^2 k_{D'_2}(x', y') \\ &\leq 8M^2 \log \left(1 + \frac{|x' - y'|}{d_{D_1}(x')} \right) \leq 8b_1 M^2 k_{D'}(x', y'). \end{aligned}$$

If $|x' - y'| \geq \frac{1}{2}d_{D_1}(x')$, then Lemma 2 shows

$$k_{D'}(x', y') \geq \log \left(1 + \frac{|x' - y'|}{d_{D'}(x')} \right) \geq \log \left(1 + \frac{1}{2b_1} \right),$$

whence

$$(3.7) \quad k_D(x, y) \leq \int_{[x, y]} \frac{|dw|}{d(w)} \leq \log 2 \leq \frac{\log 2}{\log \left(1 + \frac{1}{2b_1} \right)} k_{D'}(x', y').$$

The inequalities (3.5), (3.5) and (3.7) show that in Case 1 (3.4) holds with $M_1 = 8b_1M^2$.

Case 2. $|x - y| > \frac{1}{2}d_D(x)$.

It suffices to prove the left side inequality in (3.4) since the proof for the right one is similar. It follows from Theorem D that there exists a 2-neargeodesic γ' in D' joining x' and y' . Let $x = z_1$, and let z_2 be the first intersection point of γ with $\mathbb{S}(z_1, \frac{1}{2}d_D(z_1))$ in the direction from x to y . We let z_3 be the first intersection point of γ with $\mathbb{S}(z_2, \frac{1}{2}d_D(z_2))$ in the direction from z_2 to y . By repeating this procedure, we get a set $\{z_i\}_{i=1}^p$ of points in γ such that y is contained in $\overline{\mathbb{B}}(z_p, \frac{1}{2}d_D(z_p))$, but not in $\overline{\mathbb{B}}(z_{p-1}, \frac{1}{2}d_D(z_{p-1}))$. Obviously, $p > 1$. Hence Case 1 yields

$$\begin{aligned} k_D(x, y) &\leq \sum_{i=1}^{p-1} k_D(z_i, z_{i+1}) + k_D(z_p, y) \\ &\leq 8b_1M^2 \left(\sum_{i=1}^{p-1} k_{D'}(z'_i, z'_{i+1}) + k_{D'}(z'_p, y') \right) \\ &\leq 8b_1M^2 \ell_k(\gamma'[x', y']) \\ &\leq 16b_1M^2 k_{D'}(x', y'). \end{aligned}$$

Thus the proof of Theorem 1 is finished. \square

3.2. The proof of Theorem 2. By Theorem E, we only need to prove that f is fully (M, C) -CQH. For every subdomain D_1 in D , take $x \in D_1$ and assume that $f : D_2 = \mathbb{B}(x, d_{D_1}(x)) \rightarrow D'_2$ is φ -FQC. Then Theorem E shows that $f : D_2 \rightarrow D'_2$ is fully (M_1, C_1) -CQH, where the constants M_1, C_1 depend only on φ . Hence the similar reasoning as in the proof of Theorem 1 implies that $f : D_1 \rightarrow D'_1$ is (M, C) -CQH, where the constants M, C depend only on φ . By the arbitrariness of the subdomain D_1 in D , we see that Theorem 2 holds. \square

4. THE PROOFS OF THEOREMS 3 AND EXAMPLES 1.2 AND 1.3

To prove Theorem 3, the following definition and theorems are needed.

Definition 8. A domain D in E is called *c-uniform* in the norm metric provided there exists a constant c with the property that each pair of points z_1, z_2 in D can be joined by a rectifiable arc α in D satisfying (cf. [9, 16])

- (1) $\min_{j=1,2} \ell(\alpha[z_j, z]) \leq c d_D(z)$ for all $z \in \alpha$, and
- (2) $\ell(\alpha) \leq c |z_1 - z_2|$,

where $\ell(\alpha)$ denotes the length of α , $\alpha[z_j, z]$ the part of α between z_j and z .

In [16], Väisälä characterized uniform domains by the quasihyperbolic metric.

Theorem G. ([16, Theorem 6.16]) *For a domain D in E , the following are quantitatively equivalent:*

- (1) D is a c -uniform domain;
- (2) $k_D(z_1, z_2) \leq c' \log \left(1 + \frac{|z_1 - z_2|}{\min\{d_D(z_1), d_D(z_2)\}} \right)$ for every pair of points $z_1, z_2 \in D$;
- (3) $k_D(z_1, z_2) \leq c'_1 \log \left(1 + \frac{|z_1 - z_2|}{\min\{d_D(z_1), d_D(z_2)\}} \right) + d$ for every pair of points $z_1, z_2 \in D$.

In the case of domains in \mathbb{R}^n , the equivalence of items (1) and (3) in Theorem D is due to Gehring and Osgood [2] and the equivalence of items (2) and (3) due to Vuorinen [22].

Theorem H. ([16, Lemma 6.7]) *Suppose that G is a c -uniform domain and that $x_0 \in G$. Then $G_0 = G \setminus \{x_0\}$ is c_0 -uniform with $c_0 = c_0(c)$.*

Theorem I. ([15, Lemma 3.2]) *Let X be c -quasiconvex and let $f : X \rightarrow Y$ be a map. Then the following conditions are quantitatively equivalent:*

- (1) f is (φ, t_0) -uniformly continuous;
- (2) f is φ -uniformly continuous;
- (3) f is φ -uniformly continuous and there are $M \geq 0$ and $C \geq 0$ such that $\varphi(t) \leq Mt + C$ for all t .

From the proof of Theorem 5.7 and summary 5.11 in [15], we can get the following Lemma.

Lemma J. *Suppose that $f : D \rightarrow D'$ is a homeomorphism. If for every point $x \in D$, $f : D \setminus \{x\} \rightarrow D' \setminus \{x'\}$ is φ -solid, then f is ψ -FQC with $\psi = \psi(\varphi)$.*

Now we are ready to prove Theorem 3.

4.1. The proof of Theorem 3. By Theorem 2 and Lemma J we know that to prove the theorem we only need to prove for each $a, b \in D$ with $|a - b| \leq d_D(a)$, $f : G = \mathbb{B}(a, d_D(a)) \setminus \{b\} \rightarrow G'$ is ψ -solid with $\psi = \psi(\varphi)$. On one hand, choose $0 < t_0 < 1$ such that $\varphi(t_0) \leq \log \frac{3}{2}$. Let $x, y \in G$ be points with $k_G(x, y) \leq t_0$. Then (1.1) gives $|x' - y'| \leq \frac{1}{2} d_{G'}(x')$. Let $B_x = \mathbb{B}(x', d_{G'}(x'))$. Then by Lemma 1 we obtain

$$\begin{aligned}
k_{G'}(x', y') &\leq k_{B_x}(x', y') \leq 2 \log \left(1 + \frac{|x' - y'|}{d_{G'}(x')} \right) \\
&\leq 2j_{G'}(x', y') \leq 2\varphi(j_G(x, y)) \\
&\leq 2\varphi(k_G(x, y)).
\end{aligned}$$

Hence Theorem I yields that $f : G \rightarrow G'$ is semi-solid.

On the other hand, Theorem H show that there exists some constant $c > 1$ such that G is c -uniform. Hence Theorem G yields for each $x, y \in G$

$$k_G(x, y) \leq c' j_G(x, y) \leq c' \varphi(j_{G'}(x', y')) \leq c' \varphi(k_{G'}(x', y')),$$

where c' is a constant depending only on c . The proof of Theorem 3 is complete. \square

4.2. The proof of Example 1.2. By [2, Lemma 3], we know that conformal mapping is M -QH mapping for some constant $M \geq 1$. Hence, for each $x, y \in D$, we have

$$\frac{k_D(x, y)}{M} \leq k_{D'}(x', y') \leq M k_D(x, y).$$

Let $x', y' \in D'$ with $x' = (\frac{1}{2}, t)$ and $y' = (\frac{1}{2}, -t)$. Then

$$k_{D'}(x', y') \geq \log\left(1 + \frac{1}{t}\right),$$

and

$$j_{D'}(x', y') = \log \left(1 + \frac{|x' - y'|}{\min\{d_{D'}(x'), d_{D'}(y')\}} \right) = \log 3.$$

But

$$j_D(x, y) \geq \frac{1}{2} k_D(x, y) \geq \frac{1}{2M} k_{D'}(x', y') \geq \frac{1}{2M} \log\left(1 + \frac{1}{t}\right) \rightarrow \infty,$$

as $t \rightarrow 0$. \square

4.3. The proof of Example 1.3. By Theorem 3 that f is M^2 -QH. Let $x, y \in D$ with $x = \sqrt{2}e_1$, $y = m\sqrt{2}e_1$. Then $d_D(x) = r$, $d_D(y) = r$. Because f is locally M -bilipschitz, we get $d_{D'}(x') \geq \frac{r}{M}$ and $d_{D'}(y') \geq \frac{r}{M}$. Hence the fact “ $D' \subset \mathbb{B}(0, 2)$ ” shows that

$$j_{D'}(x', y') = \log \left(1 + \frac{|x' - y'|}{\min\{d_{D'}(x'), d_{D'}(y')\}} \right) \leq \log\left(1 + \frac{4M}{r}\right),$$

but

$$j_D(x, y) = \log \left(1 + \frac{|x - y|}{\min\{d_D(x), d_D(y)\}} \right) = \log\left(1 + \frac{\sqrt{2}(m-1)}{r}\right) \rightarrow \infty,$$

as $m \rightarrow \infty$. Hence (1.1) does not hold. \square

4.4. Remark. Theorem 1 and Theorem 2 imply that the condition “ f is M -QH (resp. φ -FQC)” is quantitatively equivalent to the condition “for every point $x \in D$, f restricted to the maximal ball $\mathbb{B}(x, d_D(x))$ is M -QH (resp. φ -FQC)”. Hence, it is natural to ask if this also holds for φ -solid.

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